Tail Dynamic Analysis of Financial Time Series
with Autoregressive Conditional Fréchet Model

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Outline

- **Intro to Extreme Value Theory**
- **Motivation**
- **Model Specification**
- **Parameter Estimation**
- **Real Data Application**
- **My work**

Two Approaches to Extreme Value

- The first method relies on deriving block maxima (minima) series generating an “Annual Maxima Series” (AMS).
- The second method relies on extracting values exceed a certain threshold (falls below a certain threshold) for some period, getting “Peak Over Threshold” (POT) values.

An Introduction to Statistical Modeling of Extreme Values
Stuart Coles, 2001.

Examples of Extreme Values

- Potential maximum daily loss across all stocks in portfolio
- Potential intra-day maximum loss of a stock
- Flood/earthquake level for civil engineering
- Extreme weather conditions
- .....

Asymptotic Models for Maxima

Let $X_1, X_2 \ldots, X_n \ldots$ be a sequence of i.i.d.-distributed random variables with distribution F, and $M_n = \max\{X_1, \ldots, X_n\}$.
In this case, we can have $Pr\{M_n < x\} = F(x)^n$

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- Estimate $F \Rightarrow F^n$
- "Central Limit Theorem" on extreme values

Why Use Asymptotic Models for Extreme Values

- Data collection/Sample
- $\hat{F}$ unreliable, worse for $\hat{F}^n$
- The variation in the maximum behaves differently from variation in the mean

Fisher–Tippett–Gnedenko (Extremal Types) Theorem

Theorem

If there exist sequences of constants $\{a_n > 0\}$ *and* $\{b_n\}$ *such that*

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = G(x)$$

where G *is a non-degenerate distribution function, then the limit distribution* G *belongs to one of the following families:*

- $G(x) = exp\{-exp[-(\frac{x-b}{a})]\}, -\infty < x < \infty$ *(Gumbel)*
- $G(x) = exp[-(\frac{x-b}{a})^{-\alpha}], x > b;$ 0 *otherwise (Fréchet)*
- $G(x) = exp\{-[-(\frac{x-b}{a})]^{\alpha}\}, x < b;$ 1 *otherwise (Weibull)*

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Generalized Extreme Value (GEV) Distribution:

$$G(x) = exp\Big\{-\Big[1 + \xi(\frac{x-\mu}{\sigma})\Big]^{-1/\xi}\Big\},$$

Generalized Pareto Distribution (GPD)

Theorem

Let $X_1, X_2 \ldots, X_n \ldots$ be a sequence of independent and identically-distributed random variables with distribution F, and $M_n = \max\{X_1, \ldots, X_n\}$. Denote arbitrary term in the X_i sequence by X, and suppose that the GEV distribution is satisfied that, for large n,

$$Pr\{M_n \leq z\} \approx G(z) = exp\Big\{ - \big[1 + \xi(\frac{x - \mu}{\sigma})\big]^{-1/\xi} \Big\}$$

Then for large enough u (threshold), the distribution function of $(X - u)$, conditional on $X > u$, is approximately

$$H(y) = 1 - (1 + \frac{\xi y}{\tilde{\sigma}})^{-1/\xi}$$

where $\tilde{\sigma} = \sigma + \xi(u - \mu)$, defined on $\{y : y > 0$ and $(1 + \xi y/\tilde{\sigma}) > 0\}$

Maxima-GEV vs. POT-GPD

Maxima-GEV

- Direct modeling of maxima, especially for high-frequency time series
- Comparable performance with POT-GPD for large sample size

POT-GPD

- More data used, more efficient
- Sensitive to threshold

Autoregressive Conditional Fréchet (AcF) Model

Modeling Maxima with Autoregressive Conditional Fréchet Model
Zifeng Zhao, Zhengjun Zhang, Rong Chen, 2018
Journal of Econometrics

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The behavior of the underlying time series may change through time, and the static approach cannot capture the dynamics of series.

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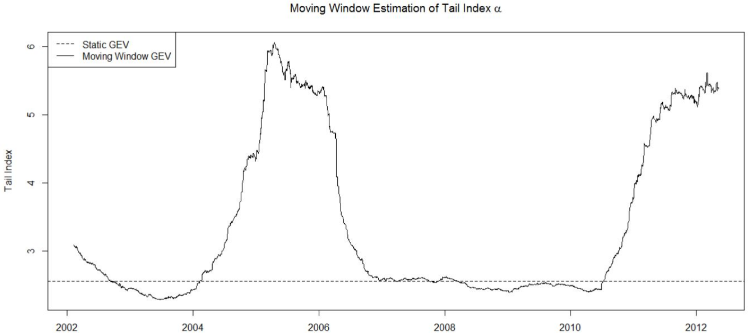

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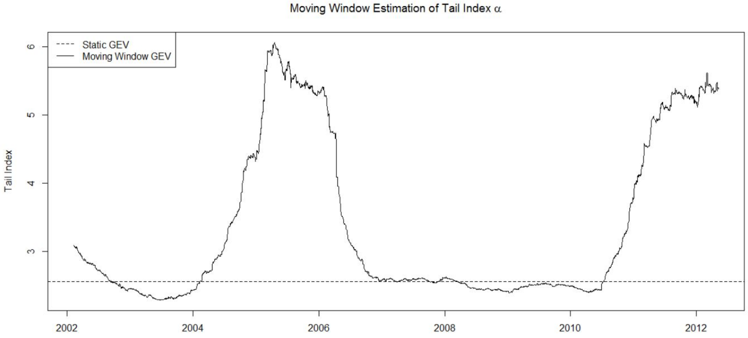

Recent studies on dynamic POT-GPD models, but little on dynamic GEV models.

Autoregressive Conditional Fréchet (AcF) Model

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Model Q_t conditionally with Fréchet distribution with parameters $(\mu_t, \sigma_t, \alpha_t) \in \mathcal{F}_{t-1} = \sigma(Q_{t-1}, Q_{t-2}, \ldots)$ where $\alpha_t = 1/\xi_t$, and the parametrization:

$$Q_t = \mu_t + \sigma_t Y_t^{1/\alpha_t}$$

where $\{Y_t\}$ is a sequence of *i.i.d.* unit Fréchet random variables.

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- The scale parameter σ_t should not be taken exactly as volatility of Q_t
  The conditional variance of Q_t depends on both on σ_t and α_t
  However, σ_t can be closely related to the volatility process of the underlying time series $\{X_{it}\}_{i=1}^{p}$

Model Specification

- μ_t is set to be constant
- $\log \sigma_t = \beta_0 + \beta_1 \log \sigma_{t-1} + \eta_1(Q_{t-1})$
- $\log \alpha_t = \gamma_0 + \gamma_1 \log \alpha_{t-1} + \eta_2(Q_{t-1})$
- $\beta_1, \gamma_1 \geq 0$
- $\eta_1(.)$ is assumed to be continuous increasing function of Q_{t-1}, and $\eta_2(.)$ is assumed to be continuous decreasing function of Q_{t-1} (clustering of extreme events)

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Further let $\eta_1(.)$, $\eta_2(.)$ to be exponential functions of the form $a_0 exp(-a_1 x)$

- Simplified version of the widely used logistic function $\frac{L}{1+a_0 exp(-a_1 x)}$
- Monotonicity, differentiability
- Implies upper bound for $\{\sigma_t\}, \{\alpha_t\}$, does not affect flexibility but facilitates numerical and techical tractability

Model Specification (Cont.)

The specific model is:

$$Q_t = \mu + \sigma_t Y_t^{1/\alpha_t}$$

$$\log \sigma_t = \beta_0 + \beta_1 \log \sigma_{t-1} - \beta_2 \exp(-\beta_3 Q_{t-1})$$

$$\log \alpha_t = \gamma_0 + \gamma_1 \log \alpha_{t-1} + \gamma_2 \exp(-\gamma_3 Q_{t-1})$$

where $\{Y_t\}$ is a sequence of *i.i.d.* unit Fréchet random variables, $0 \leq \beta_1 \neq \gamma_1 < 1, \beta_2, \beta_3, \gamma_2, \gamma_3 > 0$.

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- Can include q_1 autoregressive terms and q_2 lagged terms of Q_t
- Increase complexity and instability, but not necessarily improve model performance

Stationarity and Ergodicity

$\{\sigma_t, \alpha_t\}$ form a homogeneous Markov chain in $\mathbb{R}^2$

Theorem

For an AcF with $0 \leq \beta_1 \neq \gamma_1 < 1, \beta_2, \beta_3, \gamma_2, \gamma_3 > 0, \beta_0, \gamma_0, \mu \in \mathbb{R}$, *the latent process* $\{\sigma_t, \alpha_t\}$ *is stationary and geometrically ergodic.*

Since $\{Q_t\}$ is a coupled process of $\{\sigma_t, \alpha_t\}$, it is also stationary and ergodic.

Parameter Estimation

Denote all the parameters in the model by
$\theta = (\beta_0, \beta_1, \beta_2, \beta_3, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \mu)$
Denote the space by
$\Theta_s = \{\theta | \beta_0, \gamma_0, \mu \in \mathbb{R}, \quad \beta_1, \gamma_1 \in [0, 1), \quad \beta_2, \beta_3, \gamma_2, \gamma_3 > 0\}$

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The conditional p.d.f. of Q_t given $(\mu, \sigma_t, \alpha_t)$ is

$$f_t(\theta) = f(Q_t | \mu, \sigma_t, \alpha_t) = \alpha_t \sigma_t^{\alpha_t} (Q_t - \mu)^{-(\alpha_t + 1)} \exp\{-\sigma_t^{\alpha_t} (Q_t - \mu)^{-(\alpha_t + 1)}\}$$

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Hence, with conditional independence, the log-likelihood function with observations $\{Q_t\}_{t=1}^n$ is

$$\begin{aligned} L_n(\theta) &= \frac{1}{n} \sum_{t=1}^{n} l_t(\theta) \\ &= \frac{1}{n} \sum_{t=1}^{n} [\log \alpha_t + \alpha_t \log \sigma_t - (\alpha_t + 1) \log(Q_t - \mu) - \sigma_t^{\alpha_t} (Q_t - \mu)^{-(\alpha_t+1)}] \end{aligned}$$

$\{\sigma_t, \alpha_t\}_{t=1}^n$ can be obtained recursively.

Parameter Estimation

- True value of (σ_1, α_1) unknown

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Doesn't matter!

With $0 \leq \beta_1, \gamma_1 < 1$, the influence of initial value decay exponentially. The asymptotic distribution does not depend on (σ_1, α_1). The consistency and asymptotic normality of MLE do not depend on (σ_1, α_1).

Use static GEV estimation for initial value (burn-in)

Consistency and Asymptotic Normality of MLE

Let $\tilde{L}_n(\theta)$ be the log-likelihood based on arbitrary initial values $(\tilde{\sigma}_1, \tilde{\alpha}_1)$

Theorem (Consistency)

Assume the parameter space Θ is a compact set of Θ_s. Suppose the observations $\{Q_t\}_{t=1}^n$ are generated by a stationary and ergodic AcF with true parameter θ_0 and θ_0 is in the interior of Θ, then there exists a sequence $\hat{\theta}_n$ of local maximizer of $\tilde{L}_n(\theta)$ s.t. $\hat{\theta}_n \to_p \theta_0$ and $||\hat{\theta}_n - \theta_0|| \leq \tau_n$, where $\tau_n = O_p(n^{-r}), 0 < r < \frac{1}{2}$. Hence $\hat{\theta}_n$ is consistent.

Proposition (Asymptotic Uniqueness)

Denote $V_n = \{\theta \in \Theta | \mu \leq cQ_{n,1} + (1-c)\mu_0\}$ where $Q_{n,1} = min_{1 \leq t \leq n} Q_t$, under the conditions in previous Theorem, for any fixed $0 < c < 1$, there exists a sequence of $\hat{\theta}_n = \arg\max_{\theta \in V_n} \tilde{L}_n(\theta)$ such that $\hat{\theta}_n \to_p \theta_0$ and $||\hat{\theta}_n - \theta_0|| \leq \tau_n$, where $\tau_n = O_p(n^{-r}), 0 < r < \frac{1}{2}$, and
$\mathbb{P}(\hat{\theta}_n \text{is the unique global maximizer of } \tilde{L}_n(\theta) \text{ over } V_n) \to 1.$

This proposition gives a partial answer to the asymptotic uniqueness of MLE.V_n can be made arbitararily close to Θ_n by the fact that $Q_{n,1} \downarrow \mu_0$ a.s. and by setting c close to 1.

Consistency and Asymptotic Normality of MLE

Theorem (Asymptotic Normality)

Under the consistency theorem, we have $\sqrt{n}(\hat{\theta}_n - \theta_0) \to^d N(0, M_0^{-1})$*, where* $\hat{\theta}_n$ *is the estimator in previous theorem and* M_0 *is the Fisher Information matrix evaluated at* θ_0*. Further, the sample variance of plug-in estimated score functions* $\{\frac{\partial}{\partial\theta} l_t(\hat{\theta}_n)\}_{t=1}^n$ *is a consistent estimator of* M_0*.*

The consistency and asymptotic normality theorem show that there always exists a sequence $\hat{\theta}_n$, which is a local maximizer of $\tilde{L}_n(\theta)$, such that $\hat{\theta}_n$ is consistent and asymptotically normal regardless of the initial values $(\tilde{\sigma}_1, \tilde{\alpha}_1)$.

Simulations

- Under factor model setting

$$X_{it} = f_i(Z_{1t}, Z_{2t}, \ldots, Z_{dt}) + \sigma_{it}\epsilon_{it}$$

- With some mild conditions, the conditional distribution of maxima $Q_t = \max_{1 \le i \le p} X_{it}$ can be well approximated by a Fréchet distribution.

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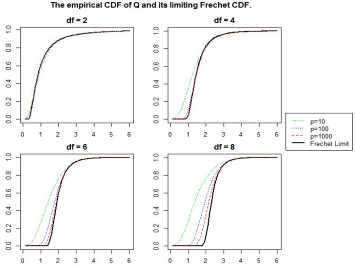
- With some mild conditions, the conditional distribution of maxima $Q_t = \max_{1 \le i \le p} X_{it}$ can be well approximated by a Fréchet distribution.
- Convergence of maxima in factor model

$X_i = \beta_i Z + \sigma_i \epsilon_i, i = 1, \ldots, p,$
$Z \sim N(0,1)$, $\beta_i \sim^{i.i.d.} U(-2,2)$, $\sigma_i \sim^{i.i.d.} \frac{1}{2}U(0.5, 1.5) + \frac{1}{2}U(0.75, 1.25)$, ϵ_i i.i.d. t-distributions with degrees of freedom ν. Repeated 1000 times.



Simulations

- Estimation for conditional VaR of maxima
- For $0 < q < 1$, the conditional Value at Risk for time t, $cVaR_t^q$ is defined as the $1 - q$ extreme quantile of Q_t, given all past information $\mathcal{F}_{t-1}$

$X_{it} = 0.009(\beta_i Z_t + \sigma_i \epsilon_{it})$
$Z_t \sim N(0,1)$, $\beta_i \sim^{i.i.d.} U(-2,2)$, $\sigma_i \sim^{i.i.d.} \frac{1}{2}U(0.5,1.5) + \frac{1}{2}U(0.75,1.25)$, ϵ_i i.i.d. t-distributions with degrees of freedom ν_t
$\log \nu_t = \gamma_0 + \gamma_1 \log \nu_{t-1} + \gamma_2 \exp(-\gamma_3 Q_{t-1})$,
$(\gamma_0, \gamma_1, \gamma_2, \gamma_3) = (-0.1, 0.9, 0.3, 5)$
Train AcF on $T_1 = 1000, 2000, 5000$, calculate next $T_2 = 100$ Q_t values.
Repeat the experiment 500 times for each combination

T_1	$\bar{q}$ $(q^0 = 0.1)$	$\bar{q}$ $(q^0 = 0.05)$	$\bar{q}$ $(q^0 = 0.01)$	mean cor.	median cor.
1000	0.095	0.049	0.012	0.871	0.928
2000	0.096	0.049	0.012	0.909	0.952
5000	0.097	0.051	0.012	0.947	0.973

Simulations

- Robust to mild dependence among ϵ_{it}

For each day, 100 ϵ_{it} are generated from 10 different multivariate t-distributions of size 10, with moderate pairwise-correlations (≤ 0.5) within each multivariate t-distribution.

T_1	$\bar{q}$ ($q^0 = 0.1$)	$\bar{q}$ ($q^0 = 0.05$)	$\bar{q}$ ($q^0 = 0.01$)	mean cor.	median cor.
1000	0.094	0.048	0.012	0.862	0.921
2000	0.097	0.047	0.011	0.876	0.936
5000	0.096	0.048	0.011	0.918	0.960

- Robust to heterogeneous ϵ_{it}

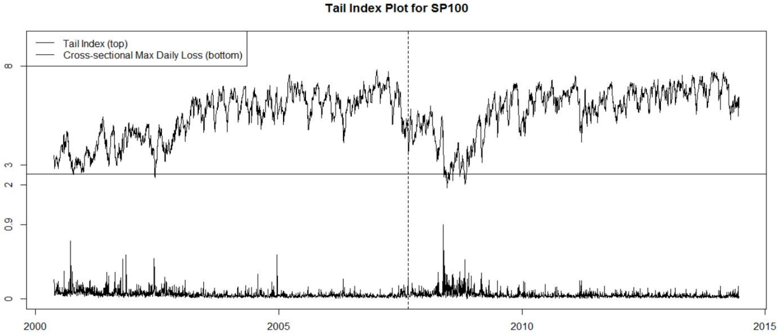
For each day, 100 ϵ_{it} are generated independently from 100 t-distributions of degree of freedom $c_i\nu_t$, c_i generated independently from $U(0.8, 1)$

T_1	$\bar{q}$ ($q^0 = 0.1$)	$\bar{q}$ ($q^0 = 0.05$)	$\bar{q}$ ($q^0 = 0.01$)	mean cor.	median cor.
1000	0.098	0.051	0.013	0.864	0.922
2000	0.097	0.050	0.012	0.905	0.953
5000	0.098	0.051	0.012	0.956	0.974

Real Data Applications

Negative daily log-returns of S&P100 Index components from January 1, 2000 to December 31, 2014 (3773 obs).

S&P100	γ_0	γ_1	γ_2	γ_3	β_0	β_1	β_2	β_3	μ
Mean	-0.068	0.890	0.328	5.33	-0.050	0.961	-0.051	6.68	-0.069
S.D.	0.014	0.013	0.063	1.27	0.006	0.004	0.0072	1.01	0.006



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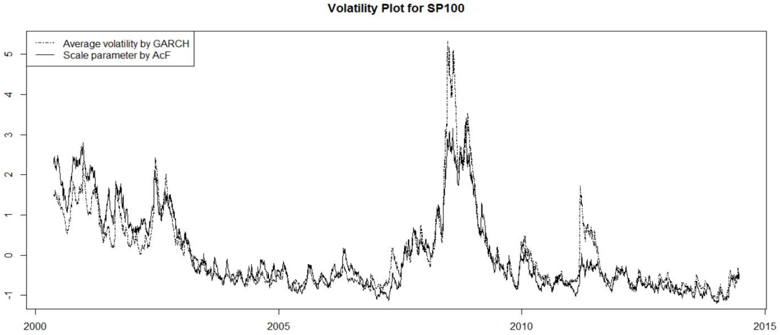


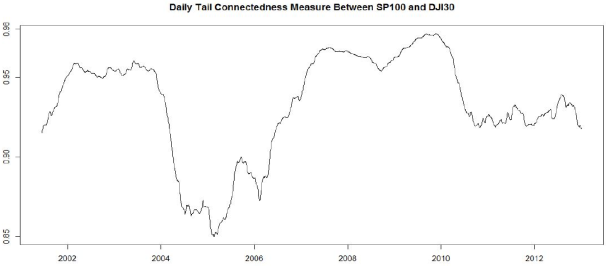
Implications for Tail-connectedness

- Let the estimated tail indexes for the S&P100 data be $\{\hat{\alpha}_t^S\}$
- Similarly applied AcF to DJI30 data, and denote the etimated tail indexes be $\{\hat{\alpha}_t^D\}$

Overall correlation between $\{\hat{\alpha}_t^S\}$ $\{\hat{\alpha}_t^D\}$ is 0.93

Let $\hat{\boldsymbol{\alpha}}_t = (\hat{\alpha}_t^S, \hat{\alpha}_t^D\ )$, let $\hat{\Sigma}$ be the sample covariance matrix of $\hat{\boldsymbol{\alpha}}_t$ based on sample $\hat{\boldsymbol{\alpha}}_{t=1}^T$.

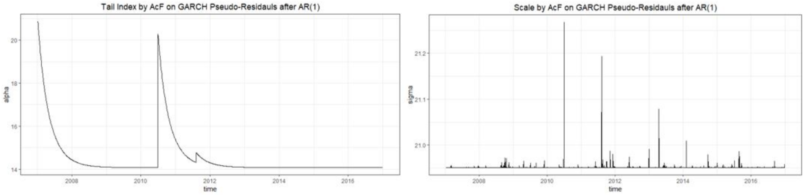
Use the ratio between the maximum eigenvalue of $\hat{\Sigma}$ and the sum of all eigenvalues as a measure of tail-connectedness. Rolling window of 36-months (about 756) days.



Post tail dynamic analysis

Apply AR(1) + GARCH(1,1) to each S&P500 component stock returns

- On daily maximal pseudo-residuals
- Test normality: Normal aggregates to Gumbel(μ, α), and exponential of Gumbel is Fréchet with location 0, scale $\exp(\mu)$, and tail $1/\alpha$.
- Apply AcF to pseudo-residuals w/o manipulation

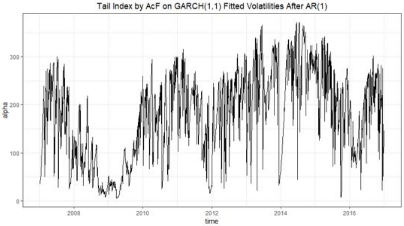


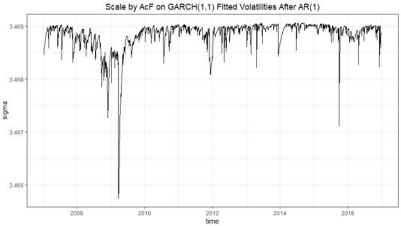
γ_1 significant while β_1 not. The aggregated maximal pseudo-residuals have phenomena of volatility clustering and clustering of extremal values, and hence the pseudo-residuals of GARCH(1,1) are not as random and independent as normally believed.

Post tail dynamic analysis

Apply AR(1) + GARCH(1,1) to each S&P500 component stock returns

- On daily maximal fitted-volatilities





The "market-wise" maximal volatilities of stock returns have its own dynamics.

Other Related Works/Further Research

- Construct pseudo-series (ghost data)
  Get the max pseudo-residual and max fitted volatility at the given time, multiply back for a pseudo-volatility. Apply AcF.
  Magnified trends
- Fama French industry classification
- Other η functions for different types of data
- Extension to light-tailed data
- ......

Thanks